Chapter 13 Mutually-inversistic model theory

Mutually-inversistic model theory includes model theory of term space and model theory of fact space, they are closely related with term space algebra and fact space algebra of mutually-inversistic abstract algebra.

13.1 Model theory of term space

Suppose the formal language of term space $L_t=\{\{p_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}\}$, where p_i are predicate constants, f_j are function constants, c_k are term constants, they are all empirical or mathematical symbols; L_t also implicitly includes logical symbols (Note that=is a mathematical symbol, not a logical symbol). The model of L_t is:

$$A_t=\langle I_0, \{p_i^A\}_{i\in I}, \{f_j^A\}_{j\in J}, \{c_k^A\}_{k\in K}\rangle,$$

where I_0 is the universe of terms, p_i^A are the interpretations of p_i in L_t, f_j^A are the interpretations of f_j, c_k^A are the interpretations of c_k. For the languages commonly used in mathematics, we use the symbols that are customary to denote the universe of terms, predicate constants, function constants, and term constants in models.

Example 13.1: Suppose we have a formal language $L_{t1}=\{\leqslant\}$, then $A_{t1}=\langle \mathbf{N}, \leqslant\rangle$ is its model. Suppose we have a formal language $L_{t2}=\{\leqslant, +, *, 0\}$, then $A_{t2}=\langle \mathbf{R}, \leqslant, +, *, 0\rangle$ is its model.

Sequence $\sigma_t=\langle a_0, a_1, a_2, \dots\rangle$ formed by choosing some elements from the universe of terms of model A_t is called the assignment of A_t, which uses elements $a_0, a_1, a_2, \dots$ from I_0 as the interpretations in model A_t of the term variables $x_0, x_1, x_2, \dots$ in the formal language L_t.

Suppose φ_t is a formula in language L_t, if φ_t is true in model A_t under assignment σ_t, then we say that φ_t is satisfied by σ_t in A_t, denoted as $A_t|=_{\sigma t}\varphi_t$, if φ_t is false, then denoted as $A_t|\neq_{\sigma t}\varphi_t$.

Example 13.2: Suppose $L_t=\{=\}$, $x=y$ is a formula in L_t, its term variables are x and y. $A_t=\langle \mathbf{N}, =\rangle$, $\sigma_{t1}=\langle 1, 1\rangle$, $\sigma_{t2}=\langle 1, 2\rangle$. Then $x=y$ is true in A_t under σ_{t1}, denoted as $A_t|=_{\sigma t1}x=y$; and $x=y$ is false in A_t under σ_{t2}, denoted as $A_t|\neq_{\sigma t2}x=y$.

If φ_t does not contain free term variables, then φ_t is called a sentence. At this time, the truth value of φ_t in model A_t is irrelevant to assignment σ_t. If φ_t is true in A_t under one assignment, then φ_t is necessarily true under other assignments. At this time, we say that φ_t is constantly true in model A_t, denoted as $A_t|=\varphi_t$. If φ_t is constantly true in any model of L_t, then we say that φ_t is logically true in L_t, denoted as $|=\varphi_t$.

Example 13.3: Suppose $L_t=\{=\}$, $x=x$ is a sentence in L_t. $X=x$ is true for all models of L_t: $A_{t1}=\langle \mathbf{N}, =\rangle$, $A_{t2}=\langle \mathbf{Z}, =\rangle$, $A_{t3}=\langle \mathbf{Q}, =\rangle$, $A_{t4}=\langle \mathbf{R}, =\rangle$, and we have $|=x=x$.

Suppose Γ_t is a set of sentences of L_t, A_t is a model of L_t. If for any sentence $\varphi_t\in\Gamma_t$, we have $A_t|=\varphi_t$, then we say that Γ_t is constantly true in A_t, denoted as $A_t|=\Gamma_t$.

Example 13.4: Suppose $L_t=\{=, +, 0\}$, its model is $A_t=\langle G, =, +, 0\rangle$. If A_t satisfies the following set of sentences:

(1) $(x+y)+z=x+(y+z)$ (associative law)

(2) $x+0=x\wedge 0+x=x$ (0 is the identity of +)

(3) $x\in G\leqslant^{-1}y\in G/\wedge^{-1}\{x+y=0\wedge y+x=0\}$ (there exists inverse element)

then we say that A_t is a group of term space, or sometimes we say that G is a group of term space. If, in addition, A_t satisfies:

(4) $x+y=y+x$ (commutative law)

then we say that A_t is an Abelian group of term space.

13.2 Model theory of fact space

Suppose we have a formal language of fact space $L_f=\{\{|\cap^{-1}, \cup|^{-1}, \subseteq^{-1}, \ldots\}, \{\cup, \cap, \sim, \ldots\}, \{\varnothing, U\}\}$. The model of L_f is $A_f=\langle I_1, \{|\cap^{-1}, \cup|^{-1}, \subseteq^{-1}, \ldots\}, \{\cup, \cap, \sim, \ldots\}, \{\varnothing, U\}\rangle$.

Sequence $\sigma_f=\langle\varnothing, \{a\}, \{b\}, \{a, b\}, \ldots\rangle$ formed by choosing some elements from universe of facts I_1 of model A_f is called the assignment of A_f, which uses elements $\varnothing$, {a}, {b}, {a, b}, … in I_1 as the interpretations in model A_f of the fact proposition variables P, Q, and R in formal language L_f.

Suppose φ_f is a formula of language L_f. If φ_f is true in model A_f under the assignment σ_f, then we say that φ_f is satisfied by σ_f in A_f, denoted as $A_f|=_{\sigma f}\varphi_f$. If φ_f is false, then denoted as $A_f|\neq_{\sigma f}\varphi_f$.

Example 13.5: Suppose $L_f=\{=^{-1}, S\}$, where S={a, b}. $P=^{-1}Q$ is a formula of L_f, its fact proposition variables are P and Q. $A_f=\langle\rho(S), =^{-1}, S\rangle$, $\sigma_{f1}=\langle\{a\}, \{a\}\rangle$, $\sigma_{f2}=\langle\{a\}, \{a, b\}\rangle$. Then $P=^{-1}Q$ is true in A_f under σ_{f1}, we have $A_f|=_{\sigma f1}P=^{-1}Q$. $P=^{-1}Q$ is false in A_f under σ_{f2}, we have $A_f|\neq_{\sigma f2}P=^{-1}Q$.

If φ_f does not contain free fact proposition variables, then φ_f is called a sentence. At this time, the truth value of φ_f in model A_f is irrelevant to assignment σ_f. If φ_f is true in A_f under one assignment, then φ_f is necessarily true under other assignments. At this time, we say that φ_f is constantly true in model A_f, denoted as $A_f|=\varphi_f$. If φ_f is constantly true in any model of L_f, then we say that φ_f is logically true in L_f, denoted as$|=\varphi_f$.

Example 13.6: Suppose $L_f=\{=^{-1}\}$. $P=^{-1}P$ is a sentence in L_f. $P=^{-1}P$ is true for model $A_f=\langle I_1, =^{-1}\rangle$ in L_f, we have $A_f|=P=^{-1}P$.

Suppose Γ_f is a set of sentences of L_f, A_f is a model of L_f. If for any sentence $\varphi_f \in \Gamma_f$, we have $A_f \models \varphi_f$, then we say that Γ_f is constantly true in A_f, denoted as $A_f \models \Gamma_f$.

Example 13.7: Suppose $L_f=\{=^{-1}, \cup, \cap, \sim, \varnothing, S\}$, where S is a non-empty set. $A_f = \langle \rho(S), =^{-1}, \cup, \cap, \sim, \varnothing, S \rangle$. If A_f satisfies the following set of sentences:

(1) $P\cup\{Q\cup R\}=^{-1}\{P\cup Q\}\cup R$, $P\cap\{Q\cap R\}=^{-1}\{P\cap Q\}\cap R$ (associative laws)

(2) $P\cup Q=^{-1}Q\cup P$, $P\cap Q=^{-1}Q\cap P$ (commutative laws)

(3) $P\cup P=^{-1}P$, $P\cap P=^{-1}P$ (idempotent laws)

(4) $P\cup\{P\cap Q\}=^{-1}P$, $P\cap\{P\cup Q\}=^{-1}P$ (absorption laws)

then we say that A_f is a lattice of fact space. If, in addition, A_f satisfies:

(5) $P\cup Q\cap R=^{-1}\{P\cup Q\}\cap\{P\cup R\}$, $P\cap\{Q\cup R\}=^{-1}P\cap Q\cup P\cap R$ (distributive laws)

then we say that A_f is a distributive lattice of fact space. If, in addition, A_f satisfies:

(6) $\sim\{P\cup Q\}=^{-1}\sim P\cap\sim Q$, $\sim\{P\cap Q\}=^{-1}\sim P\cup\sim Q$ (quasi-De Morgan's laws)

(7) $\varnothing\neq^{-1}S$, $P\cup\varnothing=^{-1}P$, $P\cup S=^{-1}S$, $P\cap\varnothing=^{-1}\varnothing$, $P\cap S=^{-1}P$, $\sim S=^{-1}\varnothing$, $\sim\varnothing=^{-1}S$ (zero-one laws)

(8) $P\cup\sim P=^{-1}S$ (law of excluded middle)

$P\cap\sim P=^{-1}\varnothing$ (non-contradiction law)

$\sim\sim P=^{-1}P$ (double complement law)

then we say that A_f is a Boolean algebra of fact space.