Chapter 38
Mutually-inversistic many-valued computer

In this chapter, we will study mutually-inversistic many-valued NAND gate and mutually-inversistic many-valued ANDORN gate, which have two schemes respectively: total ordering scheme and partial ordering scheme. Mutually-inversistic many-valued NAND gate is based on mutually-inversistic abstract algebra. Mutually-inversistic many-valued ANDORN gate is based on mutually-inversistic abstract algebra and universal matrix.

38.1 Mutually-inversistic many-valued NAND gates

38.1.1 Min-compl gate

Min-compl gate is a total ordering based *n*-valued NAND gate. *N* can be any positive integer greater than 2. We adopt *n*=3 (the most easily implementable many-valued NAND gate); i.e., the three values 0, 1, and 2. The total ordering is ⩽. The carrying set is {0, 1, 2}. The operations min and max make a complement lattice to ⩽. The complement operation ’ is: 0’=2, 1’=1, 2’=0. We take min as the AND operation, take ’(denoted by compl) as the NOT operation, thus, min-compl constitutes the total ordering based NAND gate. The mutually inverse diagram of the min-compl gate is composed of the vertices marked with “ △ ” of Fig. 38.1.

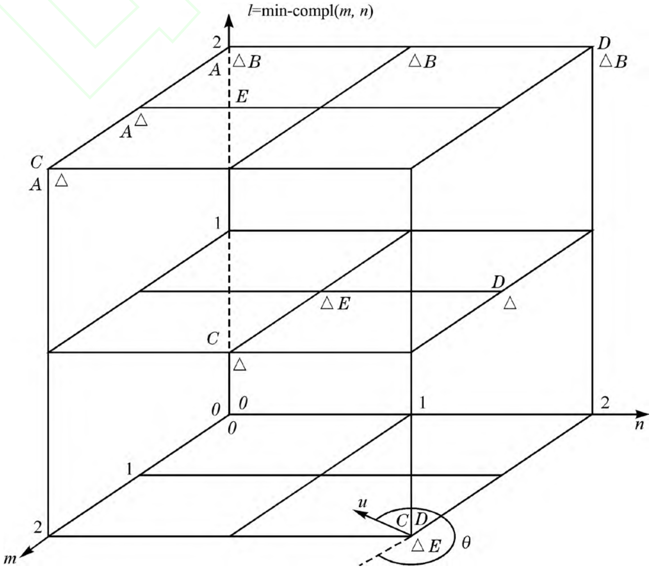


Fig. 38.1 Mutually inverse diagram for min-compl gate

In Fig. 38.1, every transaxis straight line denotes a double-sided discrete partial derivative, reveals certain algebraic property. The transaxis straight line marked with A denotes the partial derivative $\frac{\partial l}{\partial m}\Big|_{n=0}$=0, reveals the algebraic property min-compl(*m*, 0)=0'; i.e., *n*=0 is the right complement zero element. The transaxis straight line marked with B denotes the partial derivative $\frac{\partial l}{\partial n}\Big|_{m=0}$=0, reveals the algebraic property that *m*=0 is the left complement zero element. The transaxis straight line marked with C denotes the partial derivative $\frac{\partial l}{\partial n}\Big|_{m=2}$=−1, reveals the algebraic property min-compl(2, *n*)=*n*'; i.e., *m*=2 is the left complement identity element. The transaxis straight line marked with D denotes the partial derivative $\frac{\partial l}{\partial m}\Big|_{n=2}$=−1, reveals the algebraic property that *n*=2 is the right complement identity element. The transaxis straight line marked with E denotes the directional derivative $D_u$min-compl(*m*, *n*)=$\frac{\partial l}{\partial m}\cos\theta+\frac{\partial l}{\partial n}\sin\theta=\sqrt{2}$, reveals the complement idempotent law min-compl(*n*, *n*)=*n*'. If the transaxis straight lines A, C, and E are regarded as the vectors $\vec{A}$, $\vec{C}$, and $\vec{E}$, then $\vec{E}=\vec{A}+\vec{C}$. Likewise, $\vec{E}=\vec{B}+\vec{D}$.

38.1.2 GCD-compl gate

GCD-compl gate is a partial ordering based *n*-valued NAND gate. *N* adopts 2^i (*i* is any positive integer). We adopt $n=2^2=4$; i.e., the four values 1, 2, 3, and 6, which are the factors of 6. The partial ordering is dividing evenly relation. The carrying set is {1, 2, 3, 6}. The operations GCD (greatest common divisor), LCM (least common multiplier), and compl

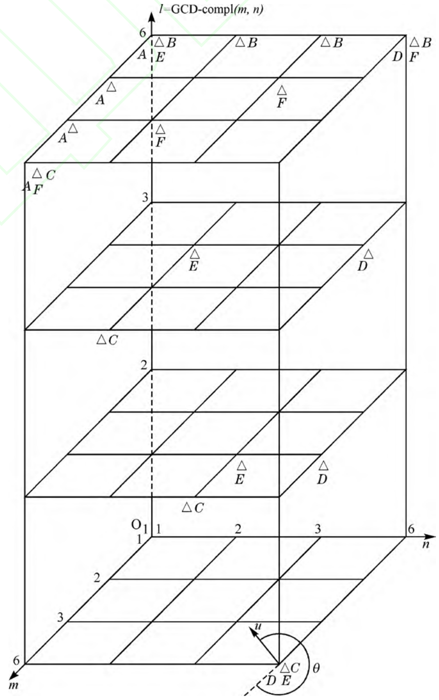


Fig. 38.2 Mutually inverse diagram for GCD-compl gate

(complement) make a Boolean algebra to the dividing evenly relation. The complement operation is: 1’=6, 2’=3, 3’=2, 6’=1. We take GCD as the AND operation, take compl as the NOT operation, thus, GCD-compl constitutes a partial ordering based NAND gate, in which 2 and 3 are not comparable, because 2 cannot divides 3 evenly, and 3 cannot divides 2 evenly. The mutually inverse diagram of GCD-compl gate is composed of the vertices marked “△” of Fig. 38.2.

In Fig. 38.2, the derivative denoted by, the algebraic property revealed by the transaxis straight lines A, B, C, D, and E are the same as those in Fig. 38.1. But Fig. 38.2 has one more transaxis straight line: line F, which denotes the partial derivative $\frac{\partial m}{\partial n}\Big|_{l=6}=-1$, reveals the algebraic property GCD-compl(*n*, *n*’)=6 (the total upper bound); i.e., the law of total upper bound. In addition to $\overrightarrow{E}=\overrightarrow{A}+\overrightarrow{C}$ and $\overrightarrow{E}=\overrightarrow{B}+\overrightarrow{D}$, Fig. 38.2 has one more sum of vectors: $\overrightarrow{F}=\overrightarrow{A}+\overrightarrow{B}$.

38.1.3 Comparison between min-compl gate and GCD-compl gate

The advantages of min-compl gate over GCD-compl gate are: min-comple gate is based on total ordering, its values adopted are natural; it can be three-valued logic, the most easily implementable many-valued logic. The advantage of GCD-compl gate over min-compl gate is: GCD-compl embraces more information: one more partial derivative, one more sum of vectors.

38.2 Mutually-inversistic many-valued ANDORN gates

38.2.1 Introduction

Two-valued ANDORN gate is $m=n_1\cdot n_2+n_3\cdot n_4$, its symbol is shown in Fig. 38.3, its truth table is shown in Table 38.1.

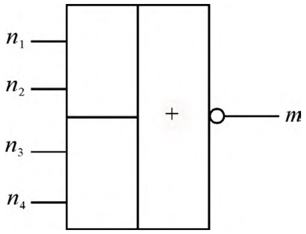


Fig. 38.3 Symbol for two-valued ANDORN gate

Table 38.1

Truth table for two-valued ANDORN gate

n_1	n_2	n_3	n_4	m
0	0	0	0	1
0	0	0	1	1
0	0	1	0	1
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	1
0	1	1	1	0
1	0	0	0	1
1	0	0	1	1
1	0	1	0	1
1	0	1	1	0
1	1	0	0	0
1	1	0	1	0
1	1	1	0	0
1	1	1	1	0

Two-valued ANDORN gate can be denoted by the 4-dimensional universal matrix shown in Fig. 38.4.

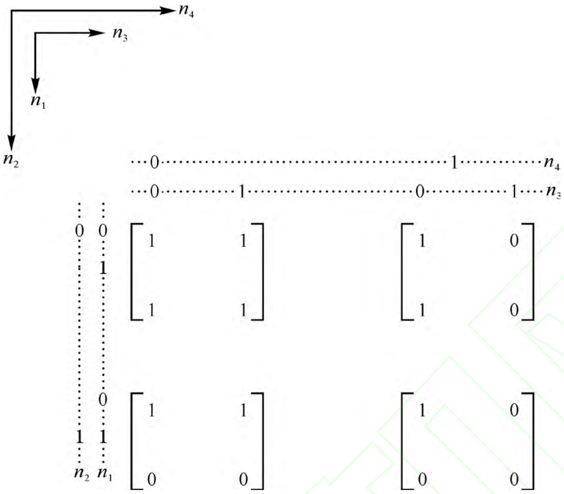


Fig. 38.4 4-dimensional universal matrix representation for two-valued ANDORN gate

The bottom left corner value 0 in the bottom left corner plane matrix of Fig. 38.4 is obtained as follows: n_1=1 and n_2=1, therefore $n_1 \wedge n_2$=1; n_3=0 and n_4=0, therefore, $n_3 \wedge n_4$=0; hence $(n_1 \wedge n_2) \vee (n_3 \wedge n_4)=1 \vee 0=1$; hence $\neg ((n_1 \wedge n_2) \vee (n_3 \wedge n_4))= \neg (1)=0$. The other values in the matrix can be similarly obtained. The universal matrix representation of two-valued ANDORN gate can be generalized to n-valued ANDORN gate.

38.2.2 Min-max-compl gate

38.2.2.1 Universal matrix representation of min-max-compl gate

Min-max-compl gate is a total ordering based n-valued ANDORN gate. N can be any positive integer greater than 2. We adopt n=3 (the most easily implementable many-valued ANDORN gate); i.e., the three values 0, 1, and 2. The total ordering is $\leqslant$. The carrying set is {0, 1, 2}. The operations min and max make a complement lattice to $\leqslant$. The complement operation ’ is: 0’=2, 1’=1, 2’=0. We take min as the AND operation, take max as the OR operation, take ’(denoted by compl) as the NOT operation, thus, min-max-compl constitutes the total ordering based ANDORN gate. Its symbol is shown in Fig. 38.5, its universal matrix representation is shown in Fig. 38.6.

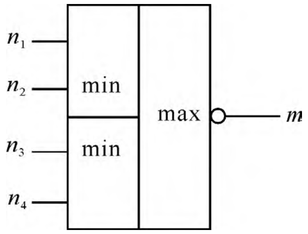
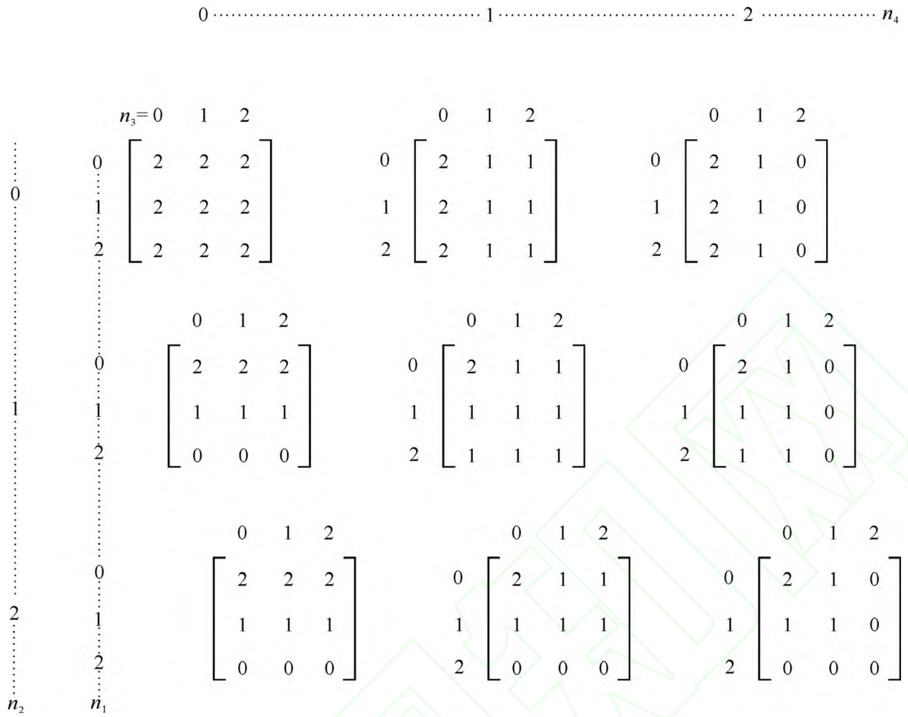


Fig. 38.5 Symbol for min-max-compl gate



$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{array}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 0 \end{array}$$

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & 0 \end{array} \quad \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 \end{array}$$

Fig. 38.6 Universal matrix representation of min-max-compl gate

Fig. 38.6 has four kinds of operations, described below.

38.2.2.2 Total lower bound type operations

Of the three values 0, 1, and 2, 0 is the total lower bound. The general form of min-max-compl operations can be denoted as

$$m=f(n_1, n_2, n_3, n_4).$$

In Fig. 38.6, there are two operations with the results being the total lower bound 0:

$$m=\mathrm{f}(2, 2, n_3, n_4)=0$$

and

$$m=\mathrm{f}(n_1, n_2, 2, 2)=0.$$

$M=\mathrm{f}(2, 2, n_3, n_4)=0$ is the bottom row of Fig. 38.6. $M=\mathrm{f}(n_1, n_2, 2, 2)=0$ is the rightmost column of Fig. 38.6. $M=\mathrm{f}(2, 2, n_3, n_4)=0$ has two partial derivatives:

$\frac{\partial \mathrm{f}}{\partial \mathrm{n3}}\Big|_{\mathrm{n1=2,\ n2=2}}=0$ (denoting that when $n_1=2$ and $n_2=2$, f does not vary with the variation of n_3, but remains 0 invariably), and $\frac{\partial \mathrm{f}}{\partial \mathrm{n4}}\Big|_{\mathrm{n1=2,\ n2=2}}=0$.

$M=\mathrm{f}(n_1, n_2, 2, 2)=0$ has also two partial derivatives:

$\frac{\partial \mathrm{f}}{\partial \mathrm{n1}}\Big|_{\mathrm{n3=2,\ n4=2}}=0$ and $\frac{\partial \mathrm{f}}{\partial \mathrm{n2}}\Big|_{\mathrm{n3=2,\ n4=2}}=0$.

38.2.2.3 Total upper bound type operations

Of the three values 0, 1, and 2, 2 is the total upper bound. In Fig. 38.6, there are four operations with the results being the total upper bound 2:

m=f(n_1, 0, n_3, 0)=2

m=f(0, n_2, 0, n_4)=2

m=f(0, n_2, n_3, 0)=2

m=f(n_1, 0, 0, n_4)=2.

Let us investigate m=f(n_1, 0, n_3, 0)=2. It is denoted by the top-left corner plane matrix. It has two partial derivatives:

$$\frac{\partial f}{\partial n1}\Big|_{n2=0,\ n4=0}=0 \text{ and } \frac{\partial f}{\partial n3}\Big|_{n2=0,\ n4=0}=0.$$

38.2.2.4 Compl type operations

In Fig. 38.6, there are 8 compl type operations:

m=f(n_1, 2, n_3, 0)=compl(n_1)

m=f(n_1, 0, n_3, 2)=compl(n_3)

m=f(2, n_2, n_3, 0)=compl(n_2)

m=f(n_1, 0, 2, n_4)=compl(n_4)

m=f(0, 2, n_3, n_4)=compl(n_4)

m=f(2, n_2, 0, n_4)=compl(n_2)

m=f(n_1, 2, 0, n_4)=compl(n_1)

m=f(0, n_2, n_3, 2)=compl(n_3).

Let us investigate m=f(n_1, 2, n_3, 0)=compl(n_1). It is denoted by the bottom-left corner plane matrix of Fig. 38.6. It has two partial derivatives:

$$\frac{\partial f}{\partial n1}\Big|_{n2=2,\ n4=0}=-1 \text{ and } \frac{\partial f}{\partial n3}\Big|_{n2=2,\ n4=0}=0.$$

38.2.2.5 Max-compl type operations

In Fig. 38.6, there are 4 max-compl type operations, which are total ordering based NOR gates:

m=f(n_1, 2, n_3, 2)=max-compl(n_1, n_3)

m=f(2, n_2, 2, n_4)=max-compl(n_2, n_4)

m=f(2, n_2, n_3, 2)=max-compl(n_2, n_3)

m=f(n_1, 2, 2, n_4)=max-compl(n_1, n_4).

Let us investigate m=f(n_1, 2, n_3, 2)=max-compl(n_1, n_3). It is denoted by the bottom-right corner plane matrix of Fig. 38.6. The plane matrix can be regarded as an operation table, the original form of which is the mutually inverse diagram shown in Fig. 38.7. The operation table is obtained by crushing the mutually inverse diagram. Fig. 38.7 is the dual diagram of Fig. 38.1, they can be studied dually. Fig. 38.7 has 5 transaxis straight lines A, B, C, D, and E. A, B, C, and D denote 4 partial derivatives, E denotes a directional derivative. They reveal 5 algebraic properties. Fig. 38.7 has two sums of vectors: $\vec{E}=\vec{A}+\vec{C}$ and $\vec{E}=\vec{B}+\vec{D}$.

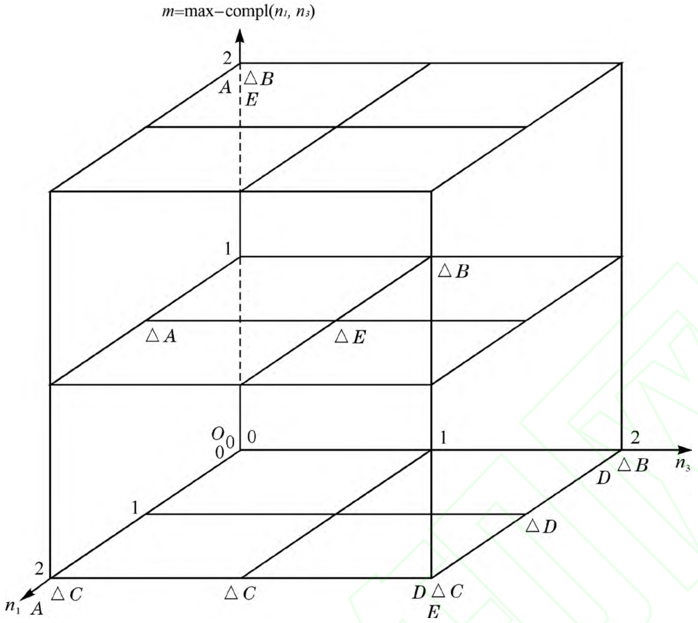


Fig. 38.7 Mutually inverse diagram for m=max-compl(n_1, n_3)

38.2.2.6 Three values fixed, the derivative of f to the fourth value

What Sections 38.2.2.2 through 38.2.2.4 discuss are actually two values fixed, the third value being arbitrary, the partial derivatives of f to the fourth value. The 4 partial derivatives A, B, C, and D discussed in Section 38.2.2.5 are actually three values fixed, the derivatives of f to the fourth value. There are more partial derivatives of the latter case. For example, the middle column of the right-middle plane matrix is a partial derivative $\frac{\partial \mathrm{f}}{\partial \mathrm{n1}}\Big|_{\mathrm{n2=1,\ n3=1,\ n4=2}}=0$.

38.2.2.7 Directional derivatives

The directional derivative E in Section 38.2.2.5 is actually two values fixed, the derivatives of f to the other two values. There are more directional derivatives like this. For example, every plane matrix at the bottom of Fig. 38.6 and every plane matrix at the right of Fig. 38.6 are a directional derivative.

38.2.3 GCD-LCM-compl gates

GCD-LCM-compl gate is a partial ordering based *n*-valued ANDORN gate. *N* adopts 2^i (i is any positive integer). We adopt $n=2^2=4$; i.e., the four values 1, 2, 3, and 6, which are the factors of 6. The partial ordering is dividing evenly relation. The carrying set is {1, 2, 3, 6}. The operations GCD (greatest common divisor), LCM (least common multiplier), and compl (complement) make a Boolean algebra to the dividing evenly relation. The complement operation is: 1'=6, 2'=3, 3'=2, 6'=1. We take GCD as the AND operation, take LCM as the OR operation, take compl as the NOT operation, thus, GCD-LCM-compl constitutes a partial ordering based ANDORN gate. The symbol of GCD-LCM-compl gate is shown in Fig. 38.8, the universal matrix representation of GCD-LCM-compl gate is shown in Fig. 38.9.

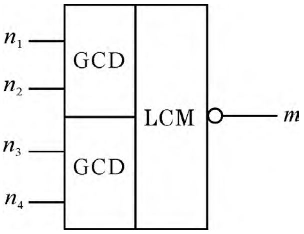


Fig. 38.8 Symbol for GCD-LCM-compl gate

$$\begin{array}{ccccc}
 & n_4 = \; 1 & 2 & 3 & 6 \\
\begin{array}{c} n_2 \\ \| \\ 1 \end{array} &
\begin{array}{cc} n_1\;\; n_3= & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}\|\\1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&6&6\\6&6&6&6\\6&6&6&6\\6&6&6&6\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&6&3\\6&3&6&3\\6&3&6&3\\6&3&6&3\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&2&2\\6&6&2&2\\6&6&2&2\\6&6&2&2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&2&1\\6&3&2&1\\6&3&2&1\\6&3&2&1\end{bmatrix}\end{array} \\
2 &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&6&6\\3&3&3&3\\6&6&6&6\\3&3&3&3\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&6&3\\3&3&3&3\\6&3&6&3\\3&3&3&3\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&2&2\\3&3&1&1\\6&6&2&2\\3&3&1&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&2&1\\3&3&1&1\\6&3&2&1\\3&3&1&1\end{bmatrix}\end{array} \\
3 &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&6&6\\6&6&6&6\\2&2&2&2\\2&2&2&2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&6&3\\6&3&6&3\\2&1&2&1\\2&1&2&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&2&2\\6&6&2&2\\2&2&2&2\\2&2&2&2\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&2&1\\6&3&2&1\\2&1&2&1\\2&1&2&1\end{bmatrix}\end{array} \\
6 &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&6&6\\3&3&3&3\\2&2&2&2\\1&1&1&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&6&3\\3&3&3&3\\2&1&2&1\\1&1&1&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&6&2&2\\3&3&1&1\\2&2&2&2\\1&1&1&1\end{bmatrix}\end{array} &
\begin{array}{cc} & \begin{matrix}1&2&3&6\end{matrix} \\ \begin{matrix}1\\2\\3\\6\end{matrix} & \begin{bmatrix}6&3&2&1\\3&3&1&1\\2&1&2&1\\1&1&1&1\end{bmatrix}\end{array}
\end{array}$$

Fig. 38.9 Universal matrix representation of GCD-LCM-compl gate

Concerning the case of two values fixed, the third value being arbitrary, the partial derivative of f to the fourth value, GCD-LCM-compl gate is the same as min-max-compo gate. Therefore, we only investigate the case of three values fixed, the partial derivative of f to the fourth value and the case of directional derivative.

First, let us investigate the plane matrix at the bottom-right corner of Fig. 38.9. It can be viewed as an operation table, denoting LCM-compl(n_1, n_3), partial ordering based NOR gate. The original form of the operation table is the mutually inverse diagram shown in Fig. 38.10. The operation table is obtained by crushing the mutually inverse diagram. In Fig. 38.10, there are 6 transaxis straight lines. The transaxis straight lines marked A, B, C, D, and F denote three values fixed, the partial derivative of f to the fourth value, in which F is absent in Fig. 38.7. The transaxis straight line E is directional derivative. Fig. 38.10 has three sum of vectors, in which $\overrightarrow{F}=\overrightarrow{C}+\overrightarrow{D}$ is absent in Fig. 38.7.

In addition to Fig. 38.10 having one more three values fixed partial derivative than Fig. 38.7, Fig. 38.9 has more three values fixed partial derivatives than Fig. 38.6 elsewhere. For example, the partial derivative $\frac{\partial f}{\partial n1}\Big|_{n2=1,\ n3=1,\ n4=2}=0$ in min-max-compl gate is one that n_2 and n_3 adopt the middle value , n_4 adopts the total upper bound, the derivative is f to n_1. As to GCD-LCM-compl gate, because there are two middle values: 2 and 3, there are two like partial derivatives. One is $\frac{\partial f}{\partial n1}\Big|_{n2=2,\ n3=2,\ n4=6}=0$, see the second column of the second plane matrix on the right. The other is $\frac{\partial f}{\partial n1}\Big|_{n2=3,\ n3=3,\ n4=6}=0$, see the third column of the third plane matrix on the right.

Fig. 38.9 has more directional derivatives than Fig. 38.6. The latter has 5 directional derivatives, while the former has 9.

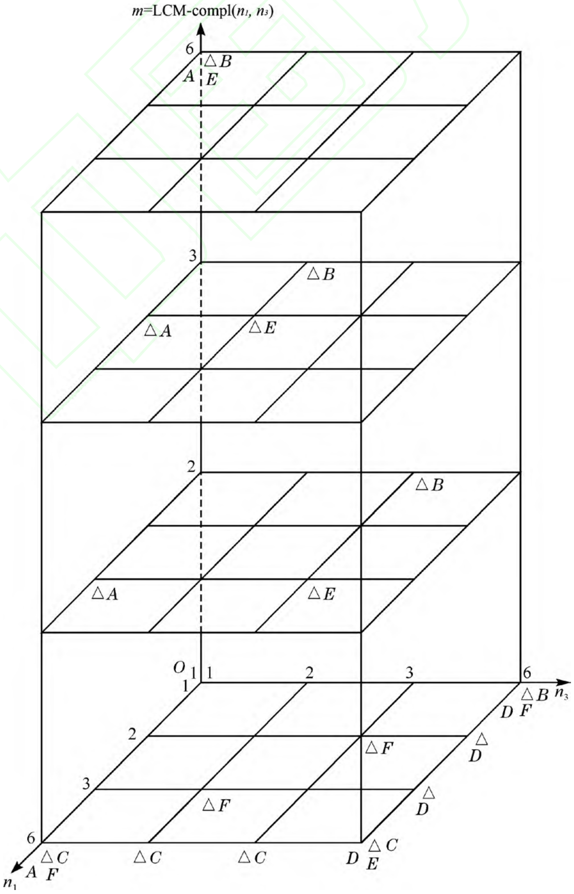


Fig. 38.10 Mutually inverse diagram for m=LCM-compl(n_1, n_3)

38.2.4 Comparison between min-max-compl gate and GCD-LCM-compl gate

The advantages of min-max-compl gate over GCD-LCM-compl gate are: min-max-comple gate is based on total ordering, its values adopted are natural; it can be three-valued logic, the most easily implementable many-valued logic. The advantage of GCD-LCM-compl gate over min-max-compl gate is: GCD-LCM-compl embraces more information: one more sum of vectors, four more directional derivatives, and several more three values fixed partial derivatives.

Someone might say: “You compare 3-valued min-max-compl gate with 4-valued GCD-LCM-compl gate. You should compare 4-valued min-max-compl gate with 4-valued GCD-LCM-compl gate.” Our answer is: “(1) If we use 4-valued min-max-compl gate to compare, we will lose the advantage that 3-valued min-max-compl gate is most easily implementable. (2) Even if we use 4-valued min-max-compl gate to compare, 4-valued GCD-LCM-compl gate still has one more partial derivative marked F, one more sum of vectors, two more directional derivatives.